ROUGH DRAFT

TOWARD A THEORY OF MARKET VALUE OF RISKY ASSETS

The objective of this paper is to lay the groundwork for a theory of market value which incorporates risk. We consider a highly idealized model of a capital market in which it is relatively easy to see how risk premiums implicit in present share prices are related to the portfolio decisions of individual investors. In a real market institutional complexities, frictions, taxes, and certain other complications which are absent in our model may have a significant effect on share prices. The aim of the paper, however, is not to present a fully developed apparatus for computing the cost of capital in practical problems. The present aim is merely:

1. to show that, under our assumptions, optimal portfolio – balancing behavior by the individual investor leads to Proposition I of the famous Modigliani-Miller paper;
2. to explore the manner in which risk affects investment value;
3. to introduce the concept of insurability. Insurable risks have a negligible effect on the cost of capital.

We will develop a mathematical definition of insurability based on the assumptions of our market model, according to which whether a risk is insurable or uninsurable is a matter of degree; nevertheless, we shall argue that it is often useful to treat risk as falling cleanly into one class or the other.

The assumptions required for the model we are about to introduce have much in common with the assumptions of the portfolio theorists (e.g., Markowitz, Tobin, Sharpe, and Farrar). The more familiar assumptions are:

1. There are no taxes.
2. There are no frictions, such as brokerage costs, to inhibit buying and selling.
3. The effect of the individual investor's decisions on prices is small enough to be disregarded.
4. Investors maximize expected utility, with primary concern for the first and second moments of the distribution of outcomes.
5. Investors are assumed averse to risk.

In addition, we assume that

6. A perfect lending market exists.
7. Investors have perfect knowledge of the market, which we interpret to mean that every investor knows
    a) present prices
    b) what every other investor knows that might have some bearing on future investment values.

If we further grant equal intelligence and equal effort to all investors, then assumption 7 is tantamount to assuming that investors agree in their forecast of future values. The emphasis in our study of the effects of risk is therefore on shifts in the market consensus over time, rather than on differences among investors at a point in time.

In a paper in the February, 1958, Review of Economic Studies James Tobin introduced the concept of dominance. Tobin envisioned an investor who was free to select his portfolio from a set of risky assets and one riskless asset - cash. He showed that one set of relative proportions of the risky assets would dominate all other possible combinations in the sense that for any given level of risk it gave the investor "the highest possible expectation of return available to him at that level of risk".* In an optimal portfolio, therefore, "the proportionate composition of the non-cash assets is independent of their aggregate share of the investment balance". An investor's attitude toward risk will be reflected in the fraction of the value of his portfolio held in cash, rather than in the proportionate composition of the non-cash assets.

Tobin's concept of dominance, slightly altered, is the starting point for this paper. In fairness to him, it must be admitted that the present development of the idea, although similar in many respects, is not entirely faithful to the original. In particular we have assumed away interest-rate risk, which was the only risk Tobin chose to consider. By focusing on interest-rate risk, Tobin sought to derive results for liquidity preference theory. Tobin's reason for limiting choice to cash and fixed-return assets was that "among these

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* p. 83.

assets cash is relatively riskless, even though in the wider context of portfolio selection, the risk of changes in purchasing power, which all monetary assets share, may be relevant to many investors". Our investor diversifies to cope with equity risk, however, rather than interest-rate risk. Tobin points out that analysis of the portfolio problem in terms of the dominant set is possible only so long as a riskless asset is available. In proposing to apply the dominance concept to the problem of choosing between fixed-return assets and equity assets, therefore, we are implicitly assuming away price-level risk as well as interest-rate risk.

The justification we offer for assuming away price-level and interest-rate risk is that, although important in other contexts, in the US economy they are both small compared to typical equity risks. The difference, which is a matter of common knowledge, is an order-of-magnitude difference. In assuming away interest-rate risk, we are also assuming away any motive on the part of the investor to hold more cash than he requires for transactions purposes. Although Tobin's condition that a riskless asset be available is not strictly met in our problem, the dominance concept is nevertheless useful in understanding the demand for equities, in which the risks are so large compared to the risks in cash and bonds that the latter seem almost riskless by comparison.

Another aspect of the present paper which diverges from the Tobin paper is the absence of positivity constraints. The individual investor is free to borrow or lend, to buy long - or sell short - as he chooses, so long as his own capital - the margin of safety for his creditors – is not wiped out.

We consider, then, a market in which there are shares in a number of equities available to investors. Like Markowitz and Tobin, we consider time broken up into arbitrary short periods within which the composition of individual portfolios is held constant. In the present paper, in which a single short time interval is under consideration, the focus is on the portfolio choices of investors at the beginning of the period, and the consequences of the choices for the prices of equity shares at the beginning of the period. From the point of view of the individual investor, the values of shares at that point in time are known (since, according to Assumption 3, his own transactions have no effect on equity prices). The values (price plus the value of distributions during the interval) at the end of the current period are unknown, hence are random variables. Assumption 7 implies that all investors share the same subjective probability distriution of the future (or terminal) value of shares. Denote the present (certain) price of a share in the (i)th equity by $v_i(0)$. Denote the future value of the (i)th equity by $v_i(1)$, and let the expected value of $v_i(1)$ be $\underline{v}_i(1)$. Then define the risk premium a_i for the (i)th equity by

$$v_i(0) = b\underline{v}_i(1) - a_i ,$$

where b is a one-period discount factor defined in terms of the lending rate r by

$$b = 1/(l+r)$$

The significance of defining risk premiums in this way becomes clear when we prove the following simple theorem.

Let x_i be the number of shares of investment (i) held in the portfolio of an investor with (equity) capital C. Then expected performance is

$$rC + 1/b \; \Sigma \; x_i \; a_i$$

In other words, the expected yield to the investor is the sum of 1) a return on his capital at the risk-free lending rate which is independent of how he invests, and 2) an expected return for risk taking which depends only on the risk taken and is independent of his capital. Unless he hoards cash, the investor will receive a return on his capital at the risk-free lending rate, no matter how he invests his money, plus a risk premium, the expected value of which depends <u>only</u> on the risk premiums for the respective investments and the position he elects to hold in each. The risk-premium concept is thus a useful one for talking about the portfolio problem under our assumptions, since, together with the uncertainty associated with a given investment, it is the relevant investment parameter.

The proof is straightforward. Expected yield P is the expected future value minus present value. Expected future value is the algebraic sum of the future values of equity shares and any debt. Suppose that the current value of the investor's equity is C and that he elects to hold x_i shares of investment (i), currently priced at $v_i(0)$. Then the difference between the value of his equity and the value of his shares must be reconciled in the lending market. The future value of the debt is the present value of the debt, appreciated at the lending rate.

Proof: Expected yield is

$$(l+r)[C - \Sigma\, x_i\, v_i(0)] + \Sigma\, x_i\, \underline{v}_i(1) - C$$

$$= rC - (l+r)\Sigma\, x_i\, v_i(0) + \Sigma\, x_i\, \underline{v}_i(1)$$

$$= rC + \Sigma\, x_i\, [\underline{v}_i(1) - (1+r)\, v_i(0)]$$

But we defined

$$v_i(0) = b\, \underline{v}_i(1) - a_i\ ,$$

whence

$$b\, \underline{v}_i(1) - v_i(0) = a_i$$

$$\underline{v}_i(1) - v_i(0)/b = \underline{v}_i(1) - (1+r)\, v_i(0)$$

$$= (1+r)\, a_i \quad .$$

Substituting in the expression for expected yield, we have

$$rC + \Sigma\, x_i\, [(1+r)\, a_i\,] = rC + 1/b\, \Sigma\, x_i\, a_i \quad .$$

Let us consider the behavior of an investor who is attempting to find an optimal balance between uncertainty and expected performance. It should be clear that

1. With reference to the individual investor's optimization problem, the level of expected performance is determined by the value of μ, defined by

$$\mu = \Sigma\, a_i\, x_i$$

2. It follows from our assumption that all investors are risk averters that for the level of expected performance which an optimal combination possesses, uncertainty is minimized.

The set of combinations with this property will dominate all other combinations in the sense of Tobin. Investors may differ, depending on their capital and attitudes toward risk, in the absolute amount of the dominant combination of risky investments they undertake, but if their (probabilistic) forecasts of future value agree, then the proportionate composition of the risky assets must be the same.

Like the portfolio theorists previously mentioned, we use variances and covariances to characterize the uncertainty in the yield of shares. Define the covariance matrix A_{ij} by

$$A_{ij} = E[(v_i(l) - \underline{v}_i(l))\ (V_j(l) - \underline{v}_j(l))]$$

where E denotes the expected value of the expression in brackets. Then the error variance σ^2 in a portfolio containing x_i shares of the (i)th equity is

$$\sigma^2 = \Sigma\ x_i\ A_{ij}\ x_j\ .$$

We shall refer occasionally to the inverse of A_{ij}, which we denote by B_{ij} :

$$\Sigma\ A_{ij}\ B_{jk} = \delta_{ik}\ .$$

To find the optimal proportions we minimize the portfolio variance subject to the constraint that expected yield

$$rC + 1/b\ \Sigma\ x_i\ a_i$$

is equal to an arbitrary constant k. For a given investor equity the constraint becomes

$$\mu = \Sigma\ a_i\ x_i\ = k\ .$$

The objective, then, is to minimize

$$\Sigma\ x_i\ A_{ij}\ x_j = \sigma^2\ ,$$

subject to

$$\mu = \Sigma\ x_i\ a_i = k\ .$$

Applying the method of Lagrange multipliers, we obtain

$$\Sigma A_{ij} x_j = \lambda/2\ a_i \quad ,$$

whence we have

$$x_j = \lambda/2\ \Sigma B_{ji} a_i \quad .$$

Substituting, we get

$$\Sigma a_j x_j - k = \lambda/2\ \Sigma a_j B_{ji} a_i - k = 0 \ ,$$

or

$$\lambda = 2k / \Sigma a_j B_{ji} a_i \ ,$$

$$x_j = \lambda/2\ \Sigma B_{ji} a_i$$

Referring back to relation

$$\Sigma A_{ij} x_j = \lambda/2\ a_i \quad ,$$

we multiply through by x_i and sum on i:

$$\Sigma x_i A_{ij} x_j = \lambda/2\ \Sigma x_i a_i = \lambda/2\ k = \sigma^2$$

The resulting expression for σ^2 enables us to write the ratio μ^2/σ^2 as

$$\mu^2/\sigma^2 = k^2/\lambda k/2 = 2k/\lambda \quad .$$

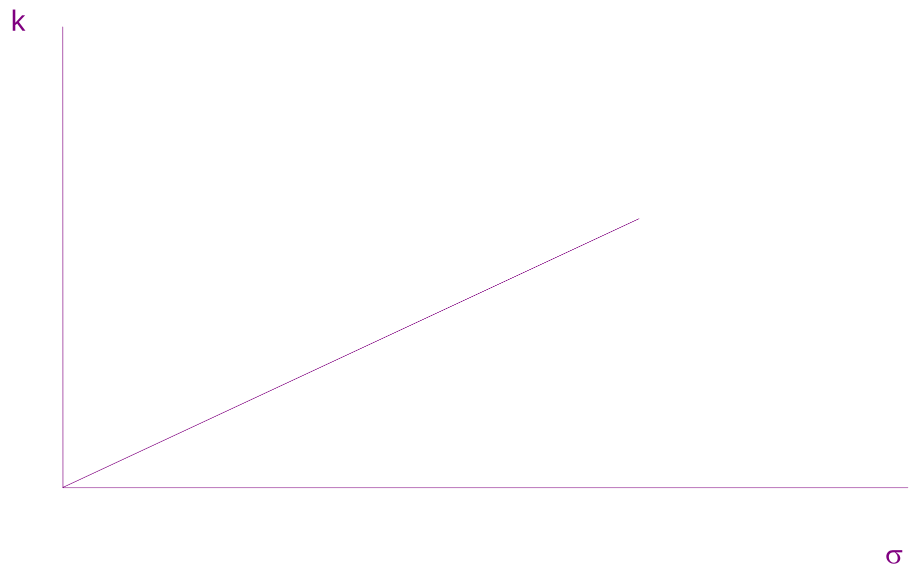
But we have

$$\lambda = 2k / \Sigma a_j B_{ji} a_i$$

so that for μ^2/σ^2 we get

$$\mu^2/\sigma^2 = \Sigma\, a_j\, B_{ji}\, a_i$$

which is independent of specified expected performance k.



All efficient combinations have the same ratio of risk premium to standard error. The efficient set is a straight line on the diagram, passing through the origin. The way in which constant - utility curves map onto the k = σ plane will depend on the investor's capital, and the lending rate, as well as his tastes. For an investor who is averse to risk, utility generally rises as one moves from southeast to northwest on the diagram. Tangency of the locus of efficient combinations (the "opportunity locus") with a utility isoquant will determine expected risk premium, μ, for the investor in question.

For the (m)th and (n)th investors, respectively, the optimal combinations are given by

$$x_{j,m} = \lambda m/2 \; \Sigma \; B_{ji} \; a_i \quad ,$$

and

$$x_{j,n} = \lambda n/2 \; \Sigma \; B_{ji} \; a_i \quad .$$

For the (j)th equity we have

$$x_{j,m} \; / \; x_{j,n} = \lambda m/\lambda n$$

The holdings of any two investors are thus identical, up to a factor of proportionality. Although the meaning of the symbols is different, except for the x_i, the preceding development is closely parallel to Tobin's.

Let X_j be the number of shares demanded by investors in the aggregate, and let Λ be defined by

$$\Lambda = \Sigma \; \lambda_n \quad .$$

Then we have

$$X_j = \Sigma \; x_{j,n} = \tfrac{1}{2} \; [\Sigma \; \lambda_n] \; \Sigma \; B_{ji} \; a_i$$

$$= \tfrac{1}{2} \; \Lambda \; \Sigma \; B_{ji} \; a_i$$

The market clearing condition is

$$X_j = \underline{X}_j$$

where $\underline{X}_j$ is the number of shares of the (j)th equity outstanding. If the market is to clear, the risk premiums a_i must satisfy

$$\tfrac{1}{2} \; \Lambda \; \Sigma \; B_{ji} \; a_j \; = \underline{X}_j$$

Solving for a_k we obtain

$$a_k = 2/\Lambda \ \Sigma \ A_{kj} \ \underline{X}_j \quad .$$

The summation is the covariance of the (k)th equity with the market as a whole.

Define k_n as the expected performance of the portfolio of the (n)th investor is

$$k_n = \Sigma \ x_{i,n} \ a_i$$

Then expected performance K for the market as a whole is

$$K = \Sigma \ k_n = \Sigma \ x_{i,n} \ a_i = \Sigma \ \underline{X}_j \ a_i$$

Using this equation and the preceding one, we can eliminate Λ from the expression for the a_k.

$$\Sigma \ \underline{X}_k \ a_k = 2/\Lambda \ \Sigma \ \underline{X}_k \ A_{kj} \ \underline{X}_j = \underline{K}$$

$$2/\underline{K} \ \Sigma \ \underline{X}_k \ A_{kj} \ \underline{X}_j$$

Hence we have for the equilibrium values of the a_k

$$a_k = \underline{K} \ \Sigma \ A_{kj} \ X_j \ / \ \Sigma \ \underline{X}_k \ A_{kj} \ \underline{X}_j$$

In our idealized equity market, therefore, the risk premium per share for the (i)th investment is proportional to the covariance of the investment with the total value of all the investments in the market.

Apparently it is a mistake to expect the risk premium to depend only on the sheer magnitude of the risk. If the uncertainty in the (i)th stock is small, or if the uncertainty is not small, but orthogonal to the market as a whole, then the risk premium will be small. The latter possibility would result in a small risk premium even for a "large" risk. This suggests that it may be useful in capital budgeting problems to distinguish between risks which by their nature can reasonably be assumed to be independent of fluctuations in the general level of the market and those which cannot. Investments which are risky only in the former sense are called insurable risks and have a cost of capital equal to the lending rate. The appraisal problem is not trivial, however, for uninsurable risks. The point is explored more fully in a subsequent paper.

It should now be clear that positivity constraints on the x_i in the portfolio problem considered first are unnecessary, since ideally the investor will hold shares in each equity in proportion to the total number of shares available in the market -- and the latter share quantities are always positive.

A second observation about the result is that, in principle at least, it suggests a way of estimating risk premiums. The A_{ij} can be estimated by taking covariances among stock-price time series, and the X_j, the number of shares of the (j)th stock outstanding, are readily available. Only K remains undetermined. A discussion of the econometric problems involved in measuring K and the A_{ij} is outside the scope of this paper.

The third point regarding the result is that it is consistent with market value in the following sense: Consider, in the simplest case, two investments with (uncertain) future values $v_1(1)$ and $v_2(1)$, and a weighted combination with (uncertain) future value v(l)

$$v(1) = \alpha_1 v_1 (1) + \alpha_2 v_2 (1) .$$

If

$$v(0) = \alpha_1 v_1 (0) + \alpha_2 v_2 (0)$$

for all v_1 and v_2 then weak linearity exists. If the covariance of v_1 with the market is $\Sigma A_{1j} X_j$ and the covariance of v_2 with the market is $\Sigma A_{2j} X_j$, then the covariance of $v = \alpha_1 v_1 + \alpha_2 v_2$ with the market is

$$\Sigma (\alpha_1 A_{1j} + \alpha_2 A_{2j}) X_j = \alpha_1 \Sigma A_{1j} X_j + \alpha_2 \Sigma A_{2j} X_j .$$

Referring back to the original definition of the risk premium, we have, as the expression for present value

$$v_i(0) = b\, v_i(1) - a_i$$

$$= b\, v_i(1) - \Sigma A_{ij} X_j (K/\Sigma X_i A_{ij} X_j).$$

Applying the expression to $v(1) = \alpha_1 v_1(1) + \alpha_2 v_2(1)$ we have

$$v(0) = b\ \underline{v}(1) - \Sigma\ (\alpha_1\ A_{1j} + \alpha_2\ A_{2j})\ X_j\ (K/\Sigma\ X_i\ A_{ij}\ X_j)$$

$$= b\ [\alpha_1\ \underline{v}_1(1) + \alpha_2\ \underline{v}_2(1)] - [\Sigma\ \alpha_1\ A_{1j}\ X_j + \Sigma\ \alpha_2\ A_{2j}\ X_j\ ]\ (K/\Sigma\ X_i\ A_{ij}\ X_j)$$

$$= \alpha_1\ v_1\ (0) + \alpha_2\ v_2\ (0)\ ,$$

hence the market-value linearity condition is satisfied. If the condition is not satisfied, then there is no assurance that the total present market value of a firm will generally be independent of how the future value is partitioned into claims.

As the following theorem shows, one-period linearity is sufficient to guarantee that proposition 1 of the Modigliani and Miller paper applies to any pattern of future earnings over time, and without restriction of those earnings to a particular risk class.

Let the future earnings for a firm at $t = 1, 2, \ldots, n$, be represented by $F(t)$. A capital structure is a set of claims $F_i(t)$ on the future earnings. Now for any given $t \neq 0$, $F(t)$ and, in general, $F_i(t)$ may be uncertain, as viewed from $t = 0$. The residual or equity claim in the set is so defined that $\Sigma F_i(t) = F(t)$, for all t. (The residual claim on earnings at a particular given point in time may of course be negative.) Define $v_i(t)$ as the value at time t of $V_i(t + 1)$, and define

$$V_i(t) = v_i(t) + F_i(t) .$$

Then for every claim $F_i(t)$ there corresponds a present value $v_i(0)$. Similarly, any other capital structure may be represented by a set of claims $F_i'(t)$ and a corresponding set of present values $v_i'(t)$. We have immediately that

$$\Sigma F_i(t) = \Sigma F_i'(t) = F(t) .$$

We shall now prove that if one-period linearity applies and $\Sigma F_i(t) = \Sigma F_i'(t)$, then $\Sigma v_i(0) = \Sigma v_i'(0)$.

If

$$\Sigma v_i(t) + F_i(t) = \Sigma v_i'(t) + F_i'(t) ,$$

then

$$\Sigma v_i(t) = \Sigma v_i'(t),$$

since by definition

$$V_i(t) = v_i(t) + F_i(t) ,$$

$$\Sigma V_i(t) = \Sigma v_i(t) + F_i(t) .$$

Now we are given

$$\Sigma F_i(t) = \Sigma F_i'(t) = F(t)$$

Using the weak linearity property we have that if

$$\Sigma V_i(t+1) = \Sigma V_i'(t+1) \ ,$$

then

$$\Sigma v_i(t) = \Sigma v_i'(t) \ .$$

Hence, adding equals to equals we have

$$\Sigma v_i(t) + F_i(t) = \Sigma v_i'(t) + F_i'(t) \ ,$$

$$\Sigma V_i(t) = \Sigma V_i'(t) \ .$$

Proof of the main theorem follows by induction, since we have shown that if

$$\Sigma V_i(t+1) = \Sigma V_i'(t+1) \ ,$$

then

$$\Sigma V_i(t) = \Sigma V_i'(t) \ ,$$

provided only that beyond some finite time T, all non-equity claims are identically zero. Then for any non-equity claims, and any t = T,

$$V_i = V_i' = 0 \ ,$$

whence we have for t = T

$$\Sigma V_i(t) = V(t) = \Sigma V_i'(t)$$

since, in the absence of other claims, the respective equity claims, hence the value at time T of the respective equity claims, must be identical.